



Separable Cubic Stochastic Operators

1. Baratov B. S.

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¹ Karshi State University, Uzbekistan

Abstract: In this paper, we study the trajectory of a separable cubic stochastic operator on a two-dimensional simplex, which naturally arises in the study of certain problems of population biology. In the simplest problem of population genetics, a biological system of a finite set consisting of n species $1, 2, \dots, n$ is considered.

Key words: cubic stochastic operator, separable cubic stochastic operator.

Introduction

There are many systems which are described by nonlinear operators. A quadratic stochastic operator (QSO) is one of the simplest nonlinear cases. A QSO has meaning of a population evolution operator and it was first introduced by Bernstein in [1]. For more than 80 years, the theory of QSOs has been developed and many papers were published (see e.g. [4]-[7]). In recent years it has again become of interest in connection with its numerous applications in many branches of mathematics, biology and physics.

Let $E = \{1, 2, \dots, m\}$ be a finite set and the set of all probability distribution on E

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_m) \in \square^m : x_i \geq 0, \text{ for any } i \text{ and } \sum_{i=1}^m x_i = 1 \right\} \quad (1)$$

be the $(m-1)$ -dimensional simplex. A QSO is a mapping defined as $V : S^{m-1} \rightarrow S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$x'_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j, \quad k \in E, \quad (2)$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^m P_{ij,k} = 1 \text{ for all } i, j \in E. \quad (3)$$

The trajectory (orbit) $\{\mathbf{x}^{(n)}\}_{n \geq 0}$, of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{(n+1)}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. This problem was solved completely for the Volterra QSO.

The operator V is called Volterra QSO, if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$, $i, j, k \in E$. For the Volterra QSO the general formula was given in [4],

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad (4)$$

where $a_{ki} = 2P_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. Moreover, $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$ for all $i, k \in E$.

In [4], the theory of Volterra QSO was developed using theory of the Lyapunov functions and tournaments. But non-Volterra QSOs were not completely studied. Because, there is no general theory that can be applied for study of non-Volterra operators.

In [7] Separable Quadratic Stochastic Operators (SQSOs) were introduced. Volterra QSO (4) has a form as SQSO, but in [8] it was proved that it coincides with a SQSO if and only if it is a linear operator.

In recent years, Cubic Stochastic Operators (CSOs) have begun to be studied, which different from quadratic operators.

In this paper we consider another class of cubic operators which we call separable cubic stochastic operators.

In Section 2, we recall the definition of CSOs and definitions and known results. In Section 3 for a SCSO defined on the two-dimensional simplex, we prove that it has three fixed points and we find conditions on parameter under which a fixed point is a repelling, attracting, or saddle point, the boundary ∂S^2 is an invariant set.

Preliminaries and known results

Separable quadratic stochastic operator. Let us recall some necessary definition and notations. In [7] Separable Quadratic Stochastic Operators were introduced as follows: The QSO (2), (3) with additional condition

$$P_{ij,k} = a_{ik} b_{jk} \text{ for all } i, j, k \in E \quad (5)$$

where $a_{ik}, b_{jk} \in \mathbb{R}$ entries of matrices $A = (a_{ik})$ and $B = (b_{jk})$ such that the conditions (3) are satisfied for the coefficients (5).

Then the QSO V corresponding to the coefficients (5) has the form.

$$x'_k = (V(\mathbf{x}))_k = (A(\mathbf{x}))_k (B(\mathbf{x}))_k, \quad (6)$$

$$\text{where } (A(\mathbf{x}))_k = \sum_{i=1}^m a_{ik} x_i, \quad (B(\mathbf{x}))_k = \sum_{j=1}^m b_{jk} x_j.$$

Definition 1. [7] The QSO (6) is called separable quadratic stochastic operator (SQSO).

From the conditions $P_{ij,k} \geq 0$ and $\sum_{k=1}^m P_{ij,k} = 1$ for all i, j it follows that the condition on matrices A and B that $a_{ik}b_{jk} \geq 0$, $AB^T = \mathbf{1}$, where B^T is the transpose of B and $\mathbf{1}$ is the matrix with all entries 1's. If $a^{(i)} = (a_{i1}, \dots, a_{im})$ is the i -th row of the matrix A and $b^{(j)} = (b_{j1}, \dots, b_{jm})$ is the j -th row of the matrix B , then from $AB^T = \mathbf{1}$ we get

$$a^{(i)}b^{(j)} = 1, \text{ for all } i, j = 1, \dots, m.$$

For a fixed j , the above condition implies that

$$A(b^{(j)})^T = (1, 1, \dots, 1). \quad (7)$$

If $\det(A) \neq 0$, then (7) gives $b^{(j)} = b^{(k)}$ for all $k, j \in E$, i.e., all rows of B are the same, therefore $\det(B) = 0$. Similarly, if $\det(B) \neq 0$, then all the rows of A must be the same, so $\det(A) = 0$.

Cubic stochastic operator. The CSO is a mapping $W : S^{m-1} \rightarrow S^{m-1}$ of the form

$$x'_l = \sum_{i,j,k=1}^m P_{ijk,l} x_i x_j x_k, \quad l \in E, \quad (8)$$

where $P_{ijk,l}$ are coefficients of heredity such that

$$P_{ijk,l} = P_{kij,l} = P_{ikj,l} = P_{kji,l} = P_{jik,l} = P_{jki,l} \geq 0, \quad \sum_{l=1}^m P_{ijk,l} = 1 \quad \forall i, j, k \in E. \quad (9)$$

and we suppose that the coefficients $P_{ijk,l}$ do not change for any permutation of i, j, k .

Note that W is a non-linear operator, and its dimension increases with m . Higher-dimensional dynamical systems are essential, but only relatively few dynamical systems have yet been analysed.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\}_{n \geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of CSO (8) is defined by $\mathbf{x}^{(n+1)} = W(\mathbf{x}^{(n)})$, where $n = 0, 1, 2, \dots$ with $\mathbf{x} = \mathbf{x}^{(0)}$. Denote by $\omega(\mathbf{x}^{(0)})$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$. Since $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is a compact set, it is follows that $\omega(\mathbf{x}^{(0)}) \neq \emptyset$. If $\omega(\mathbf{x}^{(0)})$ consists of a single point, then the trajectory converges and

$\omega(\mathbf{x}^{(0)})$ is a fixed point of the operator W . A point $\mathbf{x} \in S^{m-1}$ is called a fixed of the W if $W(\mathbf{x}) = \mathbf{x}$. Denote by $\text{Fix}(W)$ the set of all fixed points of the operator W , i.e.

$$\text{Fix}(W) = \{\mathbf{x} \in S^{m-1} : W(\mathbf{x}) = \mathbf{x}\}.$$

Let $DW(\mathbf{x}^*) = (\partial W_i / \partial x_j)(\mathbf{x}^*)$ be a Jacobian of W at the point \mathbf{x}^* .

Definition 2 ([3]): A fixed point \mathbf{x}^* is called hyperbolic if its Jacobian $DW(\mathbf{x}^*)$ has no eigenvalues on the unit circle in \mathbb{C} .

Definition 3 ([3]): A hyperbolic fixed point \mathbf{x}^* is called:

- (i) attracting if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are in the unit disk;
- (ii) repelling if all the eigenvalues of the Jacobian $DW(\mathbf{x}^*)$ are outside the closed unit disk;
- (iii) a saddle otherwise;

Main result

In this section we consider CSO (8), (9) with additional condition

$$P_{ijk,l} = a_{il}b_{jl}c_{kl}, \text{ for all } i, j, k, l \in E, \quad (11)$$

where $a_{il}, b_{jl}, c_{kl} \in \mathbb{C}$ entries of matrices $A = (a_{il})$, $B = (b_{jl})$ and $C = (c_{kl})$ such that the conditions (9) are satisfied for the coefficients (11).

Then the CSO W corresponding to the coefficients (11) has the form

$$x'_l = (W(\mathbf{x}))_l = (A(\mathbf{x}))_l (B(\mathbf{x}))_l (C(\mathbf{x}))_l \quad (12)$$

where $(A(\mathbf{x}))_l = \sum_{i=1}^m a_{il}x_i$, $(B(\mathbf{x}))_l = \sum_{j=1}^m b_{jl}x_j$ and $(C(\mathbf{x}))_l = \sum_{k=1}^m c_{kl}x_k$.

Definition 5. The CSO (12) is called separable cubic stochastic operator (SCSO).

Lemma 1. The condition (11) is sufficient for a CSO to be product of three linear operators, but the condition is not necessary.

The CSO with coefficients $P_{ijk,l}$ can be written as the product of three matrices $A = (a_{il})$, $B = (b_{jl})$ and $C = (c_{kl})$ if and only if, for any $i, j, k, l \in E$ it holds

$$P_{ijk,l} + P_{kij,l} + P_{ikj,l} + P_{kji,l} + P_{jik,l} + P_{jki,l} = a_{il}b_{jl}c_{kl} + a_{kl}b_{il}c_{jl} + a_{il}b_{kl}c_{jl} + a_{kl}b_{jl}c_{il} + a_{jl}b_{il}c_{kl} + a_{jl}b_{kl}c_{il} \quad (13)$$

condition (11) is a particular case of (13).

Let us consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 1 + \frac{a}{2} \\ 1-a & 1 & 1 \\ 1 & 1-a & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1+a & 1 \\ 1 & 1 & 1+a \\ 1 - \frac{a}{2} & 1 & 1 \end{pmatrix}, \quad (18)$$

where $a \in [-1, 1]$.

Then corresponding SCSO $W : S^2 \rightarrow S^2$ is:

$$W : \begin{cases} x'_1 = x_1 \left(x_1 + (1-a)x_2 + x_3 \right) \left(x_1 + x_2 + \left(1 - \frac{a}{2} \right) x_3 \right), \\ x'_2 = x_2 \left(x_1 + x_2 + (1-a)x_3 \right) \left((1+a)x_1 + x_2 + x_3 \right), \\ x'_3 = x_3 \left(\left(1 + \frac{a}{2} \right) x_1 + x_2 + x_3 \right) \left(x_1 + (1+a)x_2 + x_3 \right). \end{cases} \quad (19)$$

Using the equation $x_1 + x_2 + x_3 = 1$ we rewrite the operator (19) as follows

$$W : \begin{cases} x'_1 = x_1 (1 - ax_2) \left(1 - \frac{a}{2} x_3 \right), \\ x'_2 = x_2 (1 - ax_3) (1 + ax_1), \\ x'_3 = x_3 \left(1 + \frac{a}{2} x_1 \right) (1 + ax_2). \end{cases} \quad (20)$$

Evidently, that if $a = 0$ the SCSO (20) is the identity map. For this in the below, we consider the case when $a \neq 0$.

Let a face of the simplex S^2 be the set $\Gamma_\alpha = \{ \mathbf{x} \in S^2 : x_i = 0, i \notin \alpha \subset \{1, 2, 3\} \}$.

Let the set $\text{int } S^2 = \{ \mathbf{x} \in S^2 : x_1 x_2 x_3 > 0 \}$ and let the set $\partial S^2 = S^2 \setminus \text{int } S^2$ be the interior and the boundary of the simplex S^2 , respectively. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the vertexes of the two-dimensional simplex.

Theorem 1. For the SCSO W (20), the following assertions true:

(i) The face $\Gamma_{\{1,2\}}$, $\Gamma_{\{1,3\}}$, $\Gamma_{\{2,3\}}$ of the simplex S^2 are invariant sets;

(ii) $\text{Fix}(W) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$;

(iii) If $a \in [-1, 0)$, then \mathbf{e}_1 is an attracting point, \mathbf{e}_2 is a saddle point and \mathbf{e}_3 is a repelling point;

If $a \in (0, 1]$, then \mathbf{e}_1 is a repelling point, \mathbf{e}_2 is a saddle point and \mathbf{e}_3 is an attracting point.

Proof: (i) Obviously.

(ii) To find the fixed points we consider the equation $W(\mathbf{x}) = \mathbf{x}$, that is the following system of equations

$$\begin{cases} x_1 = x_1(1 - ax_2)\left(1 - \frac{a}{2}x_3\right), \\ x_2 = x_2(1 - ax_3)(1 + ax_1), \\ x_3 = x_3\left(1 + \frac{a}{2}x_1\right)(1 + ax_2). \end{cases} \quad (21)$$

(a) Let $x_1 = 0$. Then from the second equation of (21) it follows

$$x_2 = x_2 - ax_2x_3 \Rightarrow x_2 = 0 \text{ and } 1 = 1 - ax_3.$$

It is clear that if $x_2 = 0$ then we have $x_3 = 1$. If $x_2 > 0$ then from the last equation one has

$$x_3 = 0 \Rightarrow x_2 = 1.$$

We take solution $x_2 = 1$, then it follows that $x_3 = 0$. Consequently, if $x_1 = 0$, we obtain the fixed points $\mathbf{e}_2, \mathbf{e}_3$.

Similarly, in the case $x_2 = 0$ and $x_3 = 0$, we have the fixed points $\mathbf{e}_1, \mathbf{e}_3$ and $\mathbf{e}_1, \mathbf{e}_2$, respectively.

(b) Suppose that $x_1x_2x_3 \neq 0$. Then from the system (21), one has

$$\begin{cases} 1 = (1 - ax_2)\left(1 - \frac{a}{2}x_3\right), \\ 1 = (1 - ax_3)(1 + ax_1), \\ 1 = \left(1 + \frac{a}{2}x_1\right)(1 + ax_2). \end{cases} \quad (22)$$

If $a \in [-1, 0)$, then from the first equation of (22) we get

$$1 = (1 - ax_2) \left(1 - \frac{a}{2} x_3 \right) \Rightarrow \frac{a}{2} x_2 x_3 = x_2 + \frac{1}{2} x_3.$$

But from $a \in [-1, 0)$ and from the last equation it follows that $x_2 = 0$ and $x_3 = 0$, this contradicts to $x_1 x_2 x_3 \neq 0$.

If $a \in (0, 1]$, then from the third equation of (22) we have

$$1 = \left(1 + \frac{a}{2} x_1 \right) (1 + ax_2) \Rightarrow \frac{a}{2} x_1 x_2 = - \left(\frac{1}{2} x_1 + x_2 \right).$$

But from $a \in (0, 1]$ and from the last equation it follows that $x_1 = 0$ and $x_2 = 0$, this contradicts to $x_1 x_2 x_3 \neq 0$.

Consequently, we have that $\text{Fix}(W) \cap \text{int } S^2 = \emptyset$.

(iii) To find the type of fixed point of the SCSO (20), we rewrite it in the form

$$\begin{cases} x'_1 = x_1 (1 - ax_2) \left(1 - \frac{a}{2} (1 - x_1 - x_2) \right), \\ x'_2 = x_2 (1 + ax_1) (1 - a(1 - x_1 - x_2)). \end{cases} \quad (23)$$

where $(x_1, x_2) \in \{(x, y) : x, y \geq 0, 0 \leq x + y \leq 1\}$ and x_1, x_2 are the first two coordinates of a point lying in the simplex S^2 .

The Jacobian of the operator (23) at a fixed point \mathbf{e}_1 has the following eigenvalues $\lambda_1 = 1 + \frac{a}{2}$, $\lambda_2 = 1 + a$, at the \mathbf{e}_2 has the following eigenvalues $\lambda_1 = 1 - a$, $\lambda_2 = 1 + a$ and at the \mathbf{e}_3 has the following eigenvalues $\lambda_1 = 1 - a$, $\lambda_2 = 1 - \frac{a}{2}$. Therefore, it follows that:

If $a \in [-1, 0)$ then, the fixed point \mathbf{e}_1 is a attracting point, \mathbf{e}_2 is a saddle fixed point and the fixed point \mathbf{e}_3 is a repelling point.

If $a = 0$ then, fixed points \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 are non-hyperbolic points.

If $a \in (0, 1]$ then, the fixed point \mathbf{e}_1 is a repelling point, \mathbf{e}_2 is a saddle fixed point and the fixed point \mathbf{e}_3 is a attracting point.

The theorem is proved. \square

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