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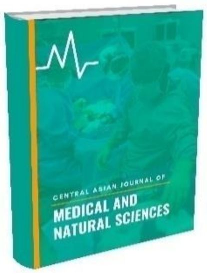


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# SOLVING HIGH-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS USING THE HYBRID BERNOULLI FUNCTION

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**Abstract:** In this research, we will apply a numerical method Dependent on hybrid Bernoulli equations to solve partial differential equations. We will also use Riemann-Liouville integral of the basic function s and use it to transform the problem to be solved into a system of nonlinear algebraic equations. Give an example to verify the ability and accuracy of the method..

Due to the importance of fractional differential equations in solving many problems, researchers have paid attention to them in recent decades. One of the most important methods used to solve these equations is the hybrid spectral clustering method .

We employed the law of impulse mass and Bernoulli's polynomials to solve this equation

$$g\left(t, f(t), D^{\beta_0} f(t), D^{\beta_1} f(t), \dots, D^{\beta_r} f(t)\right) = 0\ , 0 \leq t \leq 1, \beta_0 > \beta_1 > \cdots >> \beta_r \qquad (1)$$

where the initial conditions are

$$f^{(i)}(0) = f_i, \qquad i = 0,1, \dots, \lceil \beta_0 \rceil - 1. \qquad (2)$$

In this research, we will rely on the fractional derivative in the sens e of Caputo.

We will need Riemann-Liouville fractions to solve the fractional derivative and use numerical methods to calculate it

**Preliminaries**

Now we will review the basic concepts that we will use in this research.

Let $D^\beta$ be the Caputo derivative operator and $I^\beta$ be the Riemann -LiouvilIe fractional integral operator , fulfills the following property:

$$I^{\beta-\alpha}D^{\beta}f(t) = D^{\alpha}f(t) - \sum_{i=\lceil\alpha\rceil}^{m-1} f^{(i)}(0)\frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)}, \quad 0 < \alpha < \beta, t > 0, \tag{3}$$

The hybrid consisting of pulse mass functions and Bernoulli polynomials can be defined as follows:

$h_{nm}(t)$ $n = 1, 2, \ldots, N$ , $m = 1, 2, \ldots, M$ , on the interval $[0, 1)$ as

$$h_{nm}(t) = \begin{cases} B_m(Nt - n + 1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right) \\ 0, & \text{otherwise,} \end{cases}$$

where $n$ is the order of bloc k - pu lse func tions and $m$ is the order of Bern oulli po lynomial $Bm(t)$, which is given by the follo wing formula

$$B_m(t) = \sum_{k=0}^{m} \binom{m}{k} \alpha_{m-k} t^k,$$

in which $\alpha_m$ are the Bernoulli numbers.

**Riemann - Liouville integral of the hybrid functions**

The Riemann-Liouville equation is considered one of the most important hybrid equations using the fractional integration operator $h_{nm}(t)$, and the result was as follows:

$$I^{\beta}h_{nm}(t) = \begin{cases} b_{nm}^{\beta}(t), & \frac{n-1}{\mathrm{N}} \le t < \frac{n}{\mathrm{N}}, \\ b_{nm}^{\beta}(t) - \tilde{b}_{nm}^{\beta}(t), & \frac{n}{\mathrm{N}} \le t < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$b_{nm}^{\beta}(t) = \sum_{k=0}^{m}\sum_{r=0}^{k} \binom{m}{k}\binom{k}{r}\alpha_{m-k}\frac{N^r}{(t_f)^r}(1-n)^{k-r}$$
$$\times\left(\frac{\Gamma(r+1)t^{\beta+r}}{\Gamma(r+\beta+1)} - \frac{t^{\beta-1}\left(\frac{n-1}{N}t_f\right)^{r+1}}{(r+1)\Gamma(\beta)}\ {}_2F_1\left(r+1, 1-\beta; r+2; \frac{1}{t}\left(\frac{n-1}{N}t_f\right)\right)\right),$$

and

$$\tilde{b}_{nm}^{\beta}(t) = \sum_{k=0}^{m}\sum_{r=0}^{k} \binom{m}{k}\binom{k}{r}\alpha_{m-k}\frac{N^r}{(t_f)^r}(1-n)^{k-r}$$
$$\times\left(\frac{\Gamma(r+1)t^{\beta+r}}{\Gamma(r+\beta+1)} - \frac{t^{\beta-1}\left(\frac{n}{N}t_f\right)^{r+1}}{(r+1)\,\Gamma(\beta)}\,{}_2F_1\left(r+1,1-\beta;r+2\,;\frac{1}{t}\left(\frac{n}{N}t_f\right)\right)\right).$$

where ${}_2F_1(a,b;c;z)$ is the hyperg eometric function
Suppose that

$$\mathrm{H}(t) = [h_{10}(t),\dots,h_{1M}(t),h_{20}(t),\dots,h_{2M}(t),\dots,h_{N0}(t),\dots,h_{NM}(t)]^T.$$

Then, the Reimann-Liouville fractional integral of the basis vector $H(t)$ is obtained as

$$I^{\beta}\mathrm{H}(t) = \overline{\mathrm{H}}^{\beta}(t), \qquad (4)$$

whereas
$$\overline{\mathrm{H}}^{\beta}(t) = [I^{\beta}h_{10}(t),\dots,I^{\beta}h_{1M}(t),I^{\beta}h_{20}(t),\dots,I^{\beta}h_{2M}(t),\dots,I^{\beta}h_{N0}(t),\dots,I^{\beta}h_{NM}(t)]^T.$$

**Numerical method**

Now we will use the properties of the hybrid Bernoulli function that we mentioned previously to solve equations (1)-(2). To this end, let us assume that

$$D^{\beta_0}f(t) = A^T H(t). \qquad (5)$$

Where $A$ is the unknown coefficient vector. using (3) with $\beta = \beta_0$ and $\alpha = 0$ and (3) , also taking the initial conditions (2) into account, we get:

$$f(t) = I^{\beta_0}D^{\beta_0}f(t) + \sum_{j=0}^{\lceil\beta_0\rceil-1} f^{(j)}(0)\frac{t^j}{j!} \cong A^T\overline{H}^{\beta_0}(t) + \sum_{j=0}^{\lceil\beta_0\rceil-1} f_j\frac{t^j}{j!} \quad ,(6)$$

In the same way it works, for k = 1, 2, . . . ,p, we get

$$D^{\beta_k}f(t) = A^T\overline{H}^{\beta_0-\beta_k}(t) + \sum_{j=\lceil\beta_k\rceil}^{\lceil\beta_0\rceil-1} f_j\frac{t^j-\beta_k}{\Gamma(j-\beta_k+1)} \quad , \qquad (7)$$

By substituting (5)&(7) into (1), we have

$$g\left(t, A^T \overline{\mathrm{H}}^{\beta_0}(t) + \sum_{j=0}^{\lceil\beta_0\rceil-1} f_j \frac{t^j}{j!}, A^T H(t), A^T \overline{\mathrm{H}}^{\beta_0-\beta_1}(t) + \sum_{j=\lceil\beta_1\rceil}^{\lceil\beta_0\rceil-1} f_j \frac{t^j - \beta_1}{\Gamma(j-\beta_1+1)}, \dots, A^T \overline{\mathrm{H}}^{\beta_0-\beta_r}(t) + \sum_{j=\lceil\beta_r\rceil}^{\lceil\beta_0\rceil-1} f_j \frac{t^{j-\beta_r}}{\Gamma(j-\beta_r+1)}\right) = 0. \quad (8)$$

Suppose that

$$t_{i,j} = \frac{1}{2N}\left(\cos\left(\frac{(2j+1)\pi}{(M+1)}\right) + 2i - 1\right), \qquad i = 1,2,\dots,N, \quad j = 0,1,\dots,M,$$

which is the shifted Gauss-Chebyshev points. Now, consider $t = t_{i,j}$ in (8) a system of non linear algebraic equations in terms of the un known parameters of the vector $A$ is obtained.

We note that when solving the component system, we will obtain an approximate solution to problems (1) & (2) is given by (6).

Example. We will now provide an example that demonstrates the accuracy and efficiency of the proposed method. We will choose the following fractional differential equations

$$D^{0.5} f(t) + f(t) - \sqrt{t} - \frac{\sqrt{\pi}}{2} = 0\ , 0 < t \leq 1$$

$$f(0) = 0$$

The exact solution of this problem is $f(t) = \sqrt{t}$ . We have applied the method with different values of $M$ and $N$. Table 1 Displays the absolute error at some specified points with $M$ = 4 and $N$ = 1, 4. Moreover, the numerical solutions obtained by $N$ = 1 and $M$ = 1& 2, together with the exact solution are plotted in Figure 1.

**Table 1. Numerical results with $M$ = 4 and $N$ = 1, 4**

| $T$ | $M = 4, N = 1$ | $M = 4, N = 4$ |
|---|---|---|
| 0.2 | $5.982 \times 10^{-5}$ | $6.231 \times 10^{-6}$ |
| 0.6 | $1.163 \times 10^{-4}$ | $9.682 \times 10^{-6}$ |
| 1.0 | $4.404 \times 10^{-5}$ | $3.674 \times 10^{-6}$ |

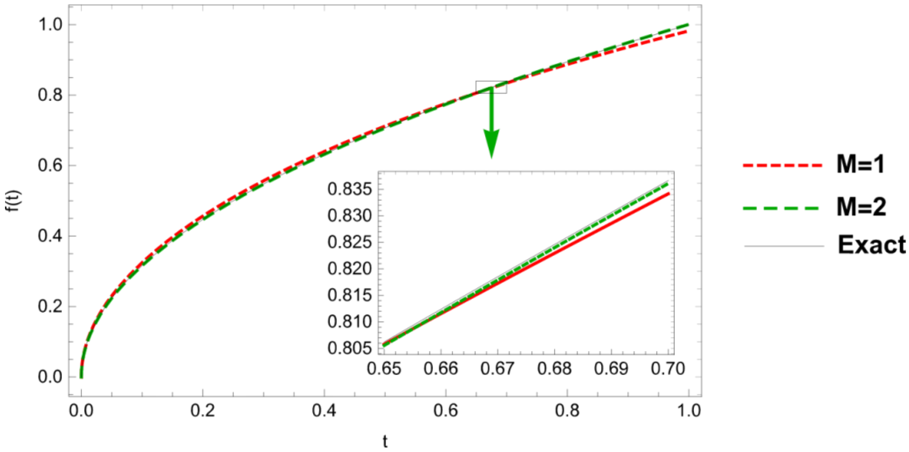


**Figure 1. Numerical solutions obtained by $N = 1$ and $M = 1$ & 2 together** with the exact solution for Example

## Conclusion

We have assumed a numerical algorithm based on the hybrid Bernoulli equation to solve High order differential equations. The algorithm relied on the fundamental properties of hybrid functions and transformed them into a nonlinear algebraic system. Then we solved an example using this algebraic system.

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